Q1 Euler's Formula

Thursday, November 11, 2010 2:54 PM

In class, we have seen that the Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$ can be used to prove many trigonometric identities.

To do this, we derived two useful expressions.

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

These relations turn any trigonomatric expressions into complex exponential functions which are usually easier to handle.

We will use the same idea here :

$$\cos A \cos B = \frac{1}{2}\left(e^{jA} + e^{-jA}\right) \times \frac{1}{2}\left(e^{jB} + e^{-jB}\right)$$

$$= \left(e^{jA} + e^{-jA}\right)\left(e^{jB} + e^{-jB}\right) \times \frac{1}{4}$$

$$= \left(\underbrace{e^{j(A+B)} + e^{-j(A+B)}}_{2\cos(A+B)} + \underbrace{e^{j(A-B)} + e^{-j(A-B)}}_{2\cos(A-B)}\right)\frac{1}{4}$$

$$= \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

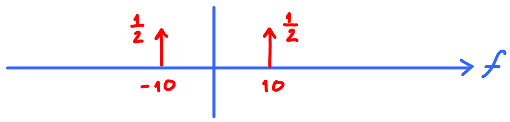
Q2 Fourier transforms of cosine functions

Wednesday, August 19, 2015 5:07 PM

First, we convert the given expressions into complex exponential functions. Then, we use the fact that $e^{j2\pi f_0 t}$ in the time domain corresponds to the delta function at $f = f_0$ in the frequency domain

(a) $\cos(\underbrace{20\pi t}_{A = 2\pi(10)t}) = \dfrac{e^{jA} + e^{-jA}}{2} = \frac{1}{2}e^{jA} + \frac{1}{2}e^{-jA} = \frac{1}{2}e^{j2\pi(10)t} + \frac{1}{2}e^{j2\pi(-10)t}$.

So, the plot of its Fourier transform is



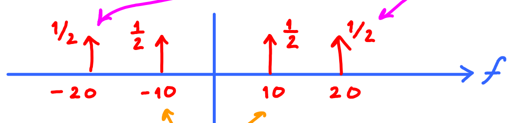
Alternatively, one may simply remember that the Fourier transform of $\cos(2\pi f_0 t)$ is simply delta functions of size $\frac{1}{2}$ at f_0 and $-f_0$.

(b) For $\cos(\underbrace{40\pi t})$, the corresponding frequencies are $\underbrace{\pm 20}$ Hz.

$2\pi f_0 t = 40\pi t$

$f_0 = 20$

So, the plot of the Fourier transform of $\cos(20\pi t) + \cos(40\pi t)$ is

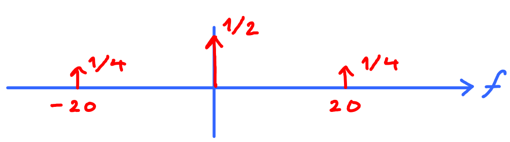


already got from part (a)

(c) $(\cos(\underbrace{20\pi t}_{A = 20\pi t = 2\pi(10)t}))^2 = (\cos A)^2 = \left(\frac{1}{2}(e^{jA} + e^{-jA})\right)^2 = \frac{1}{4}\left(e^{2jA} + 2 + e^{-2jA}\right)$

$= \frac{1}{4}e^{j2\pi(20)t} + \frac{1}{2}\underbrace{e^{j2\pi(0)t}}_{1} + \frac{1}{4}e^{j2\pi(-20)t}$

So, the plot of its Fourier transform is

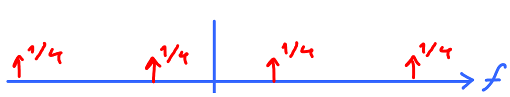


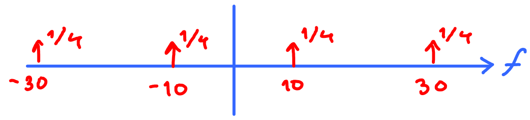
(d) $\cos(\underbrace{20\pi t}_{A = 20\pi t = 2\pi(10)t}) \times \cos(\underbrace{40\pi t}_{B = 40\pi t = 2\pi(20)t}) = \cos(A)\cos(B) = \frac{1}{2}(e^{jA} + e^{-jA})\,\frac{1}{2}(e^{jB} + e^{-jB})$

$= \frac{1}{4}\left(e^{j(A+B)} + e^{j(-A+B)} + e^{j(A-B)} + e^{j(-A-B)}\right)$

$= \frac{1}{4}\left(e^{j2\pi(30)t} + e^{j2\pi(10)t} + e^{j2\pi(-10)t} + e^{j2\pi(-30)t}\right)$

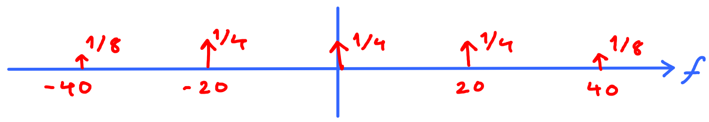
So, the plot of its Fourier transform is





(e) $$\cos^2(20\pi t) \times \cos(40\pi t) = \underbrace{\left(\frac{1}{4}e^{j2\pi(20)t} + \frac{1}{2} + \frac{1}{4}e^{j2\pi(-20)t}\right)}_{\text{from part (c)}} \times \left(\frac{1}{2}e^{j2\pi(20)t} + \frac{1}{2}e^{j2\pi(-20)t}\right)$$

$$= \frac{1}{8}e^{j2\pi(40)t} + \frac{1}{4}e^{j2\pi(20)t} + \underbrace{\frac{1}{8}e^{0} + \frac{1}{8}e^{0}}_{\frac{1}{4}e^{j2\pi(0)t}} + \frac{1}{4}e^{j2\pi(-20)t} + \frac{1}{8}e^{j2\pi(-40)t}$$



Q3 Integrals involving delta function

Wednesday, July 06, 2011 12:46 PM

(a) First, recall that $\int_A \delta(t)\,dt = \begin{cases} 1, & 0 \in A, \\ 0, & 0 \notin A \end{cases}$ In particular, $\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$

(i) $\int_{-\infty}^{\infty} 2\delta(t)\,dt = 2\int_{-\infty}^{\infty} \delta(t)\,dt = 2 \times 1 = 2.$

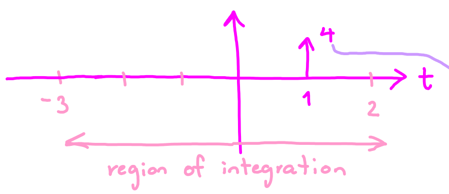
(ii) $\int_{-3}^{2} 4\delta(t-1)\,dt = 4\int_{-4}^{1} \delta(x)\,dx = 4 \times 1 = 4.$

change of variables $\begin{cases} x = t-1 \\ t = x+1 \\ dt = dx \end{cases}$

$0 \in (-4, 1)$ $\quad [\ \ ]$

Inclusion of the endpoints for this interval is not important; 0 is within the interval regardless of whether we include "-4" and "1" in to this interval.

Alternatively, consider the function $4\delta(t-1)$ graphically:



The area under the curve from -3 to 2 includes the arrow area which is 4.

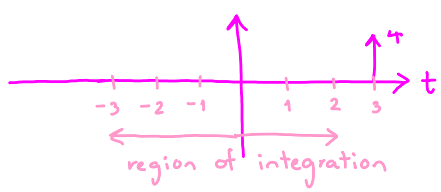
So, $\int_{-3}^{2} 4\delta(t-1)\,dt = 4.$

(iii) $\int_{-3}^{2} 4\delta(t-3)\,dt = 4\int_{-6}^{-1} \delta(x)\,dx = 4 \times 0 = 0$

change of variables $\begin{cases} x = t-3 \\ t = x+3 \\ dt = dx \end{cases}$

$0 \notin (-6, -1)$ $\quad [\ \ ]$

Alternatively, consider the function $4\delta(t-3)$ graphically:



The area under the curve from -3 to 2 does not include the arrow area.

Therefore, $\int_{-3}^{2} 4\delta(t-3)\,dt = 0.$

(b) First, recall the sampling property of the delta function

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \phi(0) \qquad (\star)$$

Therefore, $\int_{-\infty}^{\infty} \delta(t)\underbrace{e^{-j2\pi f t}}\,dt = e^{-j2\pi f t}\Big|_{t=0} = e^{0} = 1$

Therefore, $\int_{-\infty} \delta(t) \underbrace{e^{-j2\pi f t}}_{\text{set this to be } \phi(t)} dt = e^{-j2\pi f t}\Big|_{t=0} = e^0 = 1$

(c) For this part, the integration is of the form $\int_{-\infty}^{\infty} \delta(t-T) g(t)\, dt$.

We perform the "change of variables" so that the form of the expression matches the one in the sampling property:

τ is just a dummy variable here. We can simply replace it by t.

$$\int_{-\infty}^{\infty} g(t)\delta(t-T)\,dt = \int_{-\infty}^{\infty} g(\tau+T)\delta(\tau)\,d\tau = \int_{-\infty}^{\infty} g(t+T)\delta(t)\,dt$$

change of variables $\begin{cases} \tau = t-T \\ t = \tau + T \\ dt = d\tau \end{cases}$

Let $\phi(t) = g(t+T)$

$$= \int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt \overset{\star}{=} \phi(0)$$

apply the sampling property

$$= g(t+T)\Big|_{t=0} = g(T) \quad (\star\star)$$

Alternatively, consider the function $g(t)\delta(t-T)$ graphically.
Because $\delta(t-T)=0$ for $t \neq T$, the values of $g(t)$ for $t \neq T$ are not important.
We then know that $g(t)\delta(t-T) = g(T)\delta(t-T)$ which corresponds to an arrow at $t=T$ whose area is $g(T)$.

(i) $\sin(\pi t)\Big|_{t=2} = \sin(2\pi) = 0$ (use $\star\star$)

(ii) $e^{-t}\Big|_{t=-3} = e^{-(-3)} = e^3$ (use $\star\star$)

(iii) $e^{x-1}\cos\left(\frac{\pi}{2}(x-5)\right)\Big|_{x=3} = e^{3-1}\cos\left(\frac{\pi}{2}(3-5)\right) = e^2\cos(-\pi) = -e^2$ (use $\star\star$)

Note that the "x" here is just a dummy variable. It takes the role of "t" in our formula.

(d) This part has the delta function in the form $\delta(T-t)$.

We can still use the "change of variables" technique to evaluate the integral:

$$\int_{-\infty}^{\infty} g(t)\delta(T-t)\,dt = -\int_{\infty}^{-\infty} g(T-\tau)\delta(\tau)\,d\tau = \int_{-\infty}^{\infty} \underbrace{g(T-t)}_{\phi(t)}\delta(t)\,dt \overset{\star}{=} g(T-0) = g(T).$$

$\tau = T-t$ $\quad$ $t=\tau$

$(\star\star\star)$

$-\infty$ ∞

change of variables $\begin{cases} \tau = T - t \\ t = T - \tau \\ dt = -d\tau \end{cases}$

$t = \tau$, $dt = d\tau$ $\quad -\infty \quad \underbrace{}_{\emptyset(t)}$

(★★★)

Remark: Both $\int_{-\infty}^{\infty} g(t)\,\delta(t-T)\,dt$ and $\int_{-\infty}^{\infty} g(t)\,\delta(T-t)\,dt$ give $g(T)$

(i) $t^3+4\Big|_{t=1} = 1^3+4 = 1+4 = 5$ (use ★★★)

(ii) $g(2-t)\Big|_{t=3} = g(2-3) = g(-1)$ (use ★★★)

Remark: From $\int_{-\infty}^{\infty} g(t)\,\delta(T-t)\,dt = g(T)$, we get

$$\underbrace{\int_{-\infty}^{\infty} g(\tau)\,\delta(t-\tau)\,d\tau}_{(g*\delta)(t)} = g(t)$$ simply by variable renaming $\begin{pmatrix} t \to \tau \\ T \to t \end{pmatrix}$

(e) $$\int_{-2}^{2} \delta(2t)\,dt = \int_{-4}^{4} \delta(x)\,\tfrac{1}{2}\,dx = \tfrac{1}{2} \times \int_{-4}^{4} \delta(x)\,dx = \tfrac{1}{2} \times 1 = \tfrac{1}{2}.$$

change of variables $\begin{cases} x = 2t \\ t = \tfrac{1}{2}x \\ dt = \tfrac{1}{2}\,dx \end{cases}$ $\qquad 0 \in [-4,4]$

Alternatively, we know that $\delta(at) = \frac{1}{|a|}\delta(t)$. Therefore, $\delta(2t) = \frac{1}{2}\delta(t)$.

Hence, $$\int_{-2}^{2} \delta(2t)\,dt = \int_{-2}^{2} \tfrac{1}{2}\,\delta(t)\,dt = \tfrac{1}{2}\int_{-2}^{2} \delta(t)\,dt = \tfrac{1}{2} \times 1 = \tfrac{1}{2}.$$

$0 \in [-2,2]$

Q4 Time manipulation

Wednesday, July 06, 2011 12:20 PM

(a)

(i) Recall the time inversion (time reversal) operation

$g(-t)$ is the mirror image of $g(t)$ about the vertical axis.

(ii) Recall the time shifting operation:

$g(t-T)$ represents $g(t)$ time-shifted by T.

If T is positive, the shift is to the right (delay).

If T is negative, the shift is to the left (by $|T|$).

Here, $y_2(t) = g(t+6) = g(t-(-6))$.

So, $y_2(t)$ is simply $g(t)$ shifted to the left by 6 time units.

(iii) Recall the time scaling operation:

$g(at)$ is $g(t)$ compressed in time by the factor a.

for $a > 1$

So, $y_3(t) = g(3t)$ is simply $g(t)$ compressed in time by a factor of 3.

(iv) The tricky one would be $g(6-t)$.

There are two ways to think about it

① $g(t) \xrightarrow{\text{time inversion}} g(-t) \xrightarrow{\text{time shift, } T=6} g(-(t-6))$

$g(-t)$: mirror image about the vertical axis

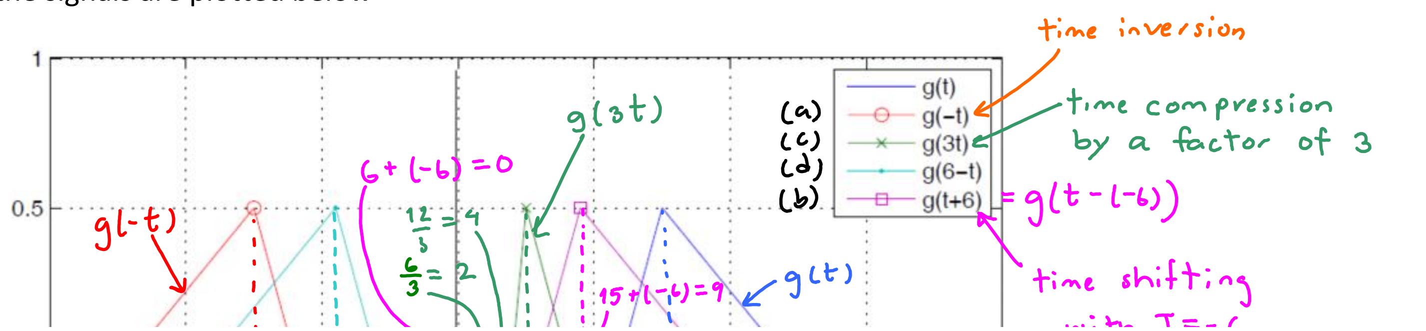
$g(-(t-6))$: shift to the right by 6

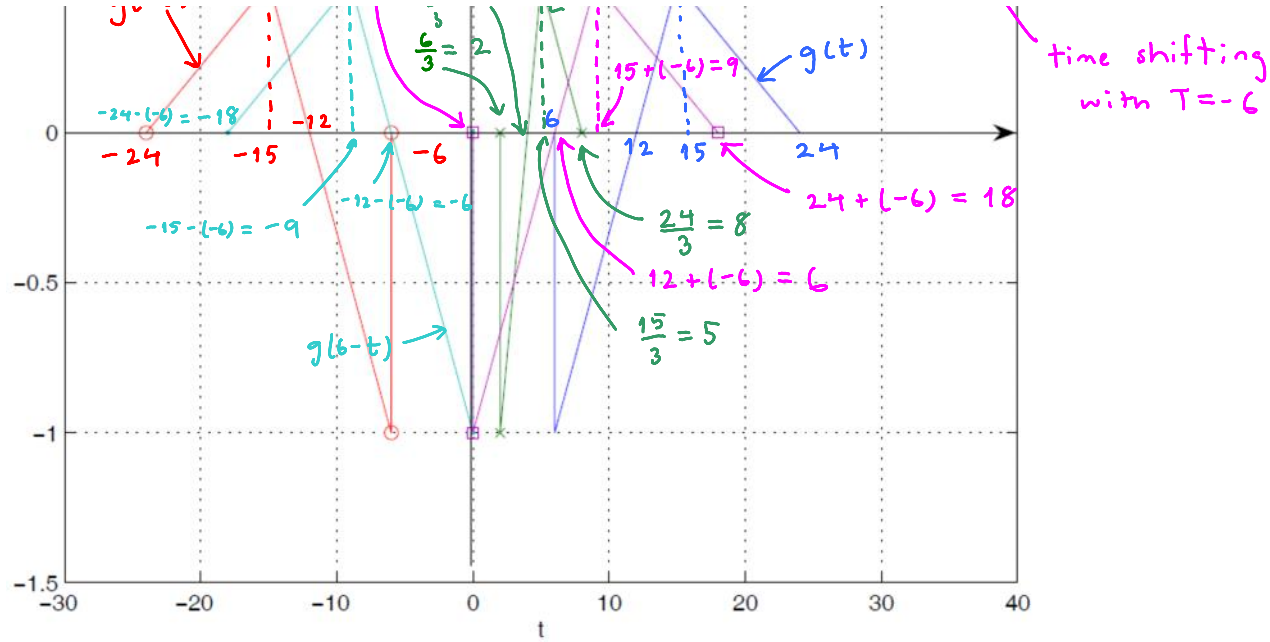
② $g(t) \xrightarrow{\text{time shift, } T=-6} g(t+6) \xrightarrow{\text{time inversion}} g(-t+6)$

$g(t+6)$: shift to the left by 6

$g(-t+6)$: mirror image of $g(t+6)$ about the vertical axis

All the signals are plotted below





(b) First, note that, for any constant m, c,

$$\int_{-\infty}^{\infty} g(mt+c)\,dt = \begin{cases} \xrightarrow{m>0} \displaystyle\int_{-\infty}^{\infty} g(\alpha)\frac{1}{m}\,d\alpha = \frac{1}{m}\int_{-\infty}^{\infty} g(\alpha)\,d\alpha \\ \xrightarrow{m<0} \displaystyle\int_{\infty}^{-\infty} g(\alpha)\frac{1}{m}\,d\alpha = -\frac{1}{m}\int_{-\infty}^{\infty} g(\alpha)\,d\alpha \end{cases} = \frac{1}{|m|}\int_{-\infty}^{\infty} g(\alpha)\,d\alpha$$

$\alpha = mt + c$

$d\alpha = m\,dt$

$dt = \frac{1}{m}\,d\alpha$

Now, for us, $\displaystyle\int_{-\infty}^{\infty} g(t)\,dt = \underbrace{-\left(\frac{1}{2}\times 1\times \overset{12-6}{6}\right)}_{\text{area under the first triangle}} + \left(\frac{1}{2}\times\frac{1}{2}\times \overset{24-12}{12}\right) = -3+3 = 0.$

Therefore, $\displaystyle\int_{-\infty}^{\infty} g(mt+c)\,dt = 0$ for any m, c.

Note:

	m	c
(i)	-1	0
(ii)	1	6
(iii)	0	0
(iv)	-1	6